

SPHERICAL ORBIT CLOSURES IN SIMPLE PROJECTIVE SPACES AND THEIR NORMALIZATION

JACOPO GANDINI

ABSTRACT. Let G be a simply connected semisimple algebraic group over an algebraically closed field k of characteristic 0 and let V be a rational simple G -module. If $G/H \subset \mathbf{P}(V)$ is a spherical orbit, set $X = \overline{G/H} \subset \mathbf{P}(V)$ its closure, then we describe the orbits of X and those of its normalization \tilde{X} . If moreover the wonderful completion of G/H is strict, then we give necessary and sufficient combinatorial conditions so that the normalization morphism $\tilde{X} \rightarrow X$ is an homeomorphism. Such conditions are trivially fulfilled if G is simply laced or if H is a symmetric subgroup.

1. INTRODUCTION.

Let G be a simply connected semisimple algebraic group over an algebraically closed field k of characteristic 0; all G -modules considered in the following will be supposed to be rational. An algebraic G -variety is said to be *spherical* if it is normal and if it contains an open B -orbit, where $B \subset G$ is a Borel subgroup; a subgroup $H \subset G$ is said to be *spherical* if the homogeneous space G/H is so: any spherical variety can thus be regarded as an open embedding of a spherical homogeneous space, namely its open G -orbit. Important classes of spherical varieties are that of toric varieties and that of symmetric varieties: toric varieties are those spherical varieties whose open orbit is an algebraic torus; symmetric varieties are those spherical varieties whose generic stabilizer H is such that $G^\sigma \subset H \subset N_G(G^\sigma)$, where $\sigma : G \rightarrow G$ is an algebraic involution and where G^σ is the set of its fixed points. Other important classes of spherical varieties are that of flag varieties and the more general one of wonderful varieties: a *wonderful variety* (of rank r) is a smooth projective G -variety having an open G -orbit and satisfying the following properties:

- the complement of the open G -orbit is the union of r smooth prime divisors having a nonempty transversal intersection;
- any orbit closure equals the intersection of prime divisors containing it.

A spherical subgroup H is said to be *wonderful* if G/H possesses a wonderful completion; by [CP] every self-normalizing symmetric subgroup is wonderful.

Many natural examples of embeddings of a spherical homogeneous space do not need to be normal. For instance, consider a simple G -module V (in which case we will call $\mathbf{P}(V)$ a *simple projective space*) with a vector v whose line $[v]$ is fixed by a spherical subgroup. Then consider the orbit $G[v] \subset \mathbf{P}(V)$, which is spherical, and take its closure $X = \overline{G[v]} \subset \mathbf{P}(V)$, which generally is not normal. The aim of this work is the study of the orbits of compactifications which arise in such a way, and as well the study of the orbits of their normalization.

In [BL], it was proved that any spherical subgroup that occurs as the stabilizer of a point in a simple projective space is wonderful; thus we can reorganize the situation as follows. Fix a wonderful variety M (whose generic stabilizer is denoted by H) and fix a divisor δ generated by its global sections; consider the simple

G -module $V = \langle Gs \rangle$ generated by its canonical section $s \in \Gamma(M, \mathcal{O}(\delta))$ and the associated projective morphism $\phi_\delta : M \rightarrow \mathbf{P}(V^*)$. Since every simple G -module containing a line fixed by H appears in such a way, the described situation is absolutely general. Set $X_\delta = \phi_\delta(M)$ and suppose without loss of generality that ϕ_δ restricts to an embedding of the open orbit $G/H \hookrightarrow X_\delta$ (this is equivalent to some combinatorial conditions on δ and on M given in [BL], see Definition 4.6). Suppose moreover that M is *strict*, i. e. that it can be embedded in a simple projective space: wonderful varieties with this property (which include the symmetric ones of [CP]) were introduced in [Pe].

Then we give necessary and sufficient combinatorial conditions on δ so that the normalization morphism $p : \tilde{X}_\delta \rightarrow X_\delta$ is bijective (Theorem 6.9). Such conditions, which involve the double links of the Dynkin diagram of G , are trivially fulfilled by a large part of strict wonderful varieties and are easily read off by the *spherical diagram* of M , which is a useful tool to represent a wonderful variety starting from the Dynkin diagram of G . Main examples of strict wonderful varieties where bijectivity fails arise from the context of *wonderful model varieties*, introduced in [L3]: the general strict case is substantially deduced from the model case.

Moreover, without any assumption on M and on δ , we describe the set of orbits of X_δ and that one of \tilde{X}_δ : for any orbit $Z \subset X_\delta$ its inverse image $p^{-1}(Z) \subset \tilde{X}_\delta$ (which is a single orbit, following a general result in [Ti]) can be nicely described starting from any orbit in M which maps on Z . Moreover we give necessary and sufficient combinatorial conditions to establish whether or not two orbits in M map to the same orbit in X_δ ; such conditions in particular imply that different orbits in X_δ are never isomorphic.

When the generic stabilizer H is a self-normalizing symmetric subgroup, compactifications in simple projective spaces were studied in [Ma] under the hypothesis that the vector $v \in V$ is fixed by the identity component of H . Under these assumptions, setting $X = \overline{G[v]} \subset \mathbf{P}(V)$, an explicit description of the G -orbits of X was given and it was proved that these orbits are equal to those of the normalization of X . Thus our results generalize those contained in [Ma].

In the case of the compactification of an adjoint group (regarded as a $G \times G$ -symmetric variety) obtained as the closure of the orbit of the identity line in the projective space $\mathbf{P}(\text{End}(V))$ (where V is a simple G -module), a complete classification of the normality and of the smoothness of such compactifications was given in [BGMR].

The paper is organized as follows. In section 2, we set notations and give preliminaries and definitions; in section 3 we give some general results about projective G -varieties in simple projective spaces having an open B -orbit and about their normalization; in section 4 we recall some recent results from [BL] about stabilizers of points in simple projective spaces and we derive some corollaries. In section 5, we describe the G -orbits of the compactifications X_δ and \tilde{X}_δ ; in section 6, we prove the theorem in the strict case, giving necessary and sufficient condition so that the normalization map is bijective; in section 7, we briefly consider the non-strict case.

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2. DEFINITIONS, NOTATIONS AND PRELIMINARIES.

Fix a simply connected semisimple algebraic group G over an algebraically closed field k of characteristic 0. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$; denote Φ the corresponding root system and $S \subset \Phi$ the corresponding set of simple roots. If $H \subset G$ is any subgroup, denote $\mathcal{X}(H)$ its character group; if V is a G -module, denote $V^{(H)}$ the set of H -eigenvectors of V , if $\chi \in \mathcal{X}(H)$, denote $V_\chi^{(H)}$ the subset of $V^{(H)}$ where H acts by χ . If $\lambda \in \mathcal{X}(B)$ is a dominant weight, we will denote V_λ the simple G -module with highest weight λ . If Λ is a lattice (by which we mean a finitely generated free \mathbf{Z} -module), then Λ^\vee denotes the dual lattice $\text{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})$ and $\Lambda_{\mathbf{Q}}$ denotes the rational vector space $\Lambda \otimes \mathbf{Q}$ generated by Λ .

If X is a spherical G -variety with open orbit G/H , let's introduce some data associated to X :

- (1) $\Lambda_X = \{B\text{-weights of rational } B\text{-eigenfunctions in } k(X)\} \simeq k(X)^{(B)}/k^*$.
- (2) $\Delta(X) = \{B\text{-stable prime divisors in } X \text{ which are not } G\text{-stable}\}$, its elements are called the *colors* of X . If $Y \subset X$ is a closed orbit, then $\Delta_Y(X)$ is the set of colors which contain Y .

Both Λ_X and $\Delta(X)$ depend only on the open orbit $G/H \subset X$ and they are the main objects of the *Luna-Vust Theory* (see [K1]), which classifies normal equivariant embeddings of a given spherical homogeneous space. A spherical variety is said to be *simple* if it possesses only one closed orbit; it is said to be *toroidal* if no color contains a closed orbit. If a spherical homogeneous space G/H possesses a complete, simple and toroidal embedding, then this is uniquely determined and it is called the *canonical embedding* of G/H ; we will denote it $M(G/H)$ and it has the following property: it dominates any simple complete embedding of G/H and it is dominated by any toroidal embedding of G/H . In general, a canonical embedding for G/H exists if and only if the index of H in its normalizer is finite, in which case H is called a *sober subgroup* of G .

If X is a simple spherical variety, then the Picard group $\text{Pic}(X)$ is freely generated by the classes $[D]$, with $D \in \Delta(X) \setminus \Delta_Y(X)$; moreover, a divisor is generated by its global sections (resp. ample) if and only if it is linearly equivalent to a linear combination of such colors with non-negative (resp. positive) coefficients (see [B1]).

Wonderful varieties are always spherical (see [L1]) and a spherical variety is wonderful if and only if it is complete, toroidal, simple and smooth. A spherical subgroup which appears as the generic stabilizer of a wonderful variety is said *wonderful*.

Suppose now M is a wonderful variety with open orbit G/H and with set of colors Δ ; suppose moreover that the center of G acts trivially on M . Let's introduce some more data we can attach to M , together with some results which can be found with more details and references in [L2] and in [BL].

- (3) $\Sigma = \{T\text{-weights of the } T\text{-module } T_z M/T_z Y\}$, where $Y \subset M$ is the closed orbit and $z \in Y^{B^-}$ is the unique fixed point (where B^- denotes the opposite Borel subgroup of B); its elements are called the *spherical roots* of M and they form a basis for the lattice $\Lambda_{G/H}$. The cardinality of Σ coincides with the *rank* of M , i. e. with the number of G -stable prime divisors of M , which are naturally in correspondence with spherical roots. If $\sigma \in \Sigma$ is a spherical root, set M^σ the corresponding G -stable prime divisor: it is a wonderful subvariety whose set of spherical roots is $\Sigma \setminus \{\sigma\}$.
- (4) $\Delta(\alpha) = \{D \in \Delta : P_\alpha D \neq D\}$ is the set of colors *moved by* α , where $\alpha \in S$ is a simple root and where P_α is the minimal parabolic associated to α . For every $\alpha \in S$, one has $0 \leq \text{card } \Delta(\alpha) \leq 2$.

- (5) $S^p = \{\alpha \in S : \Delta(\alpha) = \emptyset\}$. It coincides with the set of simple roots associated to the stabilizer of the open B -orbit, which is a parabolic subgroup and which is as well the stabilizer of the unique B -fixed point in M .
- (6) If $D \in \Delta$, set $\rho_{G/H}(D) \in \Lambda_{G/H}^\vee$ the homomorphism induced by the rational discrete valuation $\nu_D : k(M)^* \rightarrow \mathbf{Z}$ associated to D . This defines a natural pairing between colors and spherical roots

$$\begin{aligned} c(.,.) : \Delta \times \Sigma &\longrightarrow \mathbf{Z} \\ (D, \sigma) &\longmapsto c(D, \sigma) \end{aligned}$$

by $c(D, \sigma) = \langle \rho_{G/H}(D), \sigma \rangle$, which is called the *Cartan pairing* of M and which is linked to the Cartan matrix of the root system Φ as cleared in following points.

- (7) $S^a = \{\alpha \in S : \text{card } \Delta(\alpha) = 2\}$; one also has that $S^a = S \cap \Sigma$ is the set of *simple spherical roots*. Correspondingly, set $\mathbf{A} = \bigcup_{S^a} \Delta(\alpha)$ the *set of colors of type a* , where the union may be not disjoint. If $\alpha \in S^a$, set $\mathbf{A}(\alpha) = \{D_\alpha^+, D_\alpha^-\}$; then

$$c(D_\alpha^+, \sigma) + c(D_\alpha^-, \sigma) = \langle \alpha^\vee, \sigma \rangle$$

for every spherical root σ . Moreover, if $\sigma \in \Sigma$ is a spherical root and if $D \in \mathbf{A}$, then it holds $c(D, \sigma) = 1$ if and only if $\sigma \in S$ and $D \in \Delta(\sigma)$.

- (8) Set $S^{2a} = \{\alpha \in S : 2\alpha \in \Sigma\}$; if $\alpha \in S^{2a}$, then $\text{card } \Delta(\alpha) = 1$. Correspondingly, set $\Delta^{2a} = \bigcup_{S^{2a}} \Delta(\alpha)$ the *set of colors of type $2a$* , where the union is always disjoint. If $\alpha \in S^{2a}$, set $\Delta(\alpha) = \{D_\alpha\}$; then

$$c(D_\alpha, \sigma) = \langle \alpha^\vee, \sigma \rangle / 2$$

for every spherical root σ .

- (9) Set $S^b = S \setminus (S^p \cup S^a \cup S^{2a})$; if $\alpha \in S^b$, then $\text{card } \Delta(\alpha) = 1$. Correspondingly, set $\Delta^b = \bigcup_{S^b} \Delta(\alpha)$ the *set of colors of type b* . If $\alpha, \beta \in S^b$, then one has $\Delta(\alpha) = \Delta(\beta)$ if and only if α and β are orthogonal and $\alpha + \beta \in \Sigma$. If $\alpha \in S^b$, set $\Delta(\alpha) = \{D_\alpha\}$; then

$$c(D_\alpha, \sigma) = \langle \alpha^\vee, \sigma \rangle$$

for every spherical root σ .

- (10) One has $\Delta = \mathbf{A} \cup \Delta^{2a} \cup \Delta^b$, and the union is always disjoint.
- (11) $\mathcal{S} = (\Sigma, S^p, \mathbf{A})$ is the *spherical system* of M . Here \mathbf{A} has to be thought of as an abstract set, together with the pairing $c : \mathbf{A} \times \Sigma \rightarrow \mathbf{Z}$, i. e. as a multisubset of the dual lattice $(\mathbf{Z}\Sigma)^\vee = \Lambda_{G/H}^\vee$. This is the combinatorial datum which expresses a wonderful variety: each wonderful variety is uniquely determined by its spherical system (see [Lo]). There is also an abstract combinatorial definition of spherical system (see [L2]), introduced in order to obtain a classification of wonderful varieties. Anyway, it is still an open question (*Luna's conjecture*) whether or not abstract spherical systems classify wonderful varieties: while the “uniqueness part” was proved in [Lo], the “existence part” is still open, even if it has been proved in many cases. There is a very useful way to represent spherical systems by means of *spherical diagrams*, obtained adding information to the Dynkin diagram of Φ (see [L2] and [BL]).
- (12) If $\Sigma' \subset \Sigma$ is a subset of spherical roots, then the *localization at Σ'* of M is the variety

$$M_{\Sigma'} = \bigcap_{\sigma \in \Sigma \setminus \Sigma'} M^\sigma :$$

it is a wonderful variety whose spherical system is $\mathcal{S}' = (\Sigma', S^p, \mathbf{A}')$, where $\mathbf{A}' = \bigcup_{\alpha \in S \cap \Sigma'} \Delta(\alpha)$.

- (13) Let K be a spherical subgroup and let $N_G(K)$ be its normalizer, acting on the right on G/K by $n \cdot gK = gnK$. Consider the induced action of $N_G(K)$ on $\Delta(G/K)$: the kernel of such action is called the *spherical closure* of K ; if K coincides with its spherical closure, then it is called *spherically closed*. Spherically closed subgroups are always wonderful (see [K3]); a wonderful variety is said to be *spherically closed* if its generic stabilizer is so. Recently, it has been proved that a spherical group is spherically closed if and only if it occurs as the stabilizer of a point in a simple projective space (see [BL]).
- (14) M is said to be *strict* if every orbit stabilizer is self-normalizing; equivalently, we will say also that H is strict. A wonderful variety is strict if and only if it can be embedded in a simple projective space (see [Pe]).
- (15) Consider the following sets of spherical roots

$$\Sigma_\ell^D(M) = \left\{ \sigma \in \Sigma \setminus S : \begin{array}{l} \text{there exists a rank 1 wonderful variety} \\ \text{whose spherical system is } (2\sigma, S^p, \emptyset) \end{array} \right\},$$

$$\Sigma_\ell^S(M) = \{ \sigma \in S \cap \Sigma : c(D_\alpha^+, \sigma) = c(D_\alpha^-, \sigma) \ \forall \sigma \in \Sigma \};$$

set $\Sigma_\ell(M) = \Sigma_\ell^D(M) \cup \Sigma_\ell^S(M)$ the *set of loose spherical roots*. Loose spherical roots of the first kind are easily described, they are those of the following shapes (where $S = \{\alpha_1, \dots, \alpha_n\}$ and simple roots are labeled as in Bourbaki):

- spherical roots of the shape $\alpha_{i+1} + \dots + \alpha_{i+r}$, with support of type B_r and with $\alpha_{i+r} \in S^p$;
- spherical roots of the shape $2\alpha_{i+1} + \alpha_{i+2}$, with support of type G_2 .

For every $\sigma \in \Sigma_\ell(M)$, one can define a G -equivariant automorphism $\gamma(\sigma) \in \text{Aut}_G(M)$ of order 2 which fixes pointwise the G -stable divisor M^σ associated to σ . If $\sigma \in \Sigma_\ell^D(M)$, then $\gamma(\sigma)$ acts trivially on Δ , while if $\sigma \in \Sigma_\ell^S(M)$, then $\gamma(\sigma)$ exchanges D_σ^+ and D_σ^- and acts trivially on $\Delta \setminus \Delta(\sigma)$. Moreover, such elements commute and generate $\text{Aut}_G(M)$ (see [Lo]).

By the natural identification $\text{Aut}_G(M) = N_G(H)/H$, we obtain

- H is self-normalizing if and only if $\Sigma_\ell(M) = \emptyset$;
- H is spherically closed if and only if $\Sigma_\ell^D(M) = \emptyset$;
- H is strict if and only if $S \cap \Sigma = \emptyset$ and $\Sigma_\ell(M) = \emptyset$.

In particular, if $S \cap \Sigma = \emptyset$, then H is self-normalizing if and only if it is spherically closed if and only if it is strict.

3. NORMALIZATION OF THE CLOSURE OF A SPHERICAL ORBIT IN A SIMPLE PROJECTIVE SPACE.

Let V be a simple G -module and $G/H \simeq Gx_0 \subset \mathbf{P}(V)$ a spherical orbit; set $X = \overline{Gx_0}$. Since X contains finitely many B -orbits ([K2], Corollary 2.6), every G -orbit in X is spherical; denote $Y \subset X$ the unique closed orbit.

Let $p : \tilde{X} \rightarrow X$ be the normalization of X ; then \tilde{X} is a simple and complete spherical variety with the same open orbit of X whose orbits are naturally in bijection with those of X :

Proposition 3.1 ([Ti], Proposition 1). *The normalization morphism $p : \tilde{X} \rightarrow X$ is bijective on the level of G -orbits.*

If $Z \subset X$ is an orbit, in the following we will denote by Z' the corresponding orbit $p^{-1}(Z) \subset \tilde{X}$. Fix Z and fix base points $z_0 \in Z$ and $z'_0 \subset p^{-1}(z_0) \subset Z'$ so that we have isomorphisms

$$Z' \simeq G/K', \quad Z \simeq G/K,$$

where $K' \subset K$ are the stabilizers in G of z_0 and z'_0 respectively.

Let's recall a result which will be useful in the following:

Theorem 3.2 ([BP], Proposition 5.1 and Corollary 5.2). *Let H be a spherical subgroup of G . Then*

- (i) *The algebraic group $N_G(H)/H$ is diagonalizable; moreover, if H^0 is the identity component of H , then $N_G(H) = N_G(H^0)$.*
- (ii) *If B is any Borel subgroup such that BH is open in G , then $N_G(H)$ equals the right stabilizer of BH .*

Coming back to our situation, then we obtain:

Corollary 3.3. *K' is normal in K with finite index; in particular K/K' is a finite diagonalizable group.*

Proof. Since p is a finite morphism, it preserves dimensions of orbits: so we have $\dim(K') = \dim(K)$. Then the inclusion $K' \subset K$ implies $(K')^0 = K^0$ and, by Theorem 3.2, we obtain

$$N_G(K') = N_G((K')^0) = N_G(K^0) = N_G(K);$$

so we have

$$(K')^0 = K^0 \subset K' \subset K \subset N_G(K) = N_G(K').$$

Therefore $[K : K^0] < \infty$ implies that $[K : K'] < \infty$, while $K/K' \subset N_G(K')/K'$ implies that K/K' is a diagonalizable group. \square

If $P' \subset P$ are the stabilizers of the B -fixed point respectively in \tilde{X} and in X , then they are parabolic subgroups of the same dimension, so they are equal. Therefore Y and Y' are isomorphic; from now on we will denote both of them with the same letter Y .

Lemma 3.4. *Let $K' \subset K$ be two spherical subgroups of G , with K' normal in K ; fix a Borel subgroup B such that BK' is open in G and consider the projection $\pi : G/K' \rightarrow G/K$. Then $\pi^{-1}(BK/K) = BK'/K'$ and $\pi^* : \Lambda_{G/K} \rightarrow \Lambda_{G/K'}$ identifies $\Lambda_{G/K}$ with a sublattice of $\Lambda_{G/K'}$ such that*

$$\Lambda_{G/K'} / \Lambda_{G/K} \simeq \mathcal{X}(K/K').$$

Proof. Let's show that $BK' = BK$, which implies the first claim. Take $k \in K$ and consider $BK'k$ and BK' : since they are both open subsets, their intersection is non-empty. Take then $a, b \in B$ and $m, n \in K'$ such that $amk = bn$: since $k^{-1}m^{-1}k \in K'$, then we get $k = a^{-1}bn(k^{-1}m^{-1}k) \in BK'$.

As for the second claim, the equality $BK = BK'$ together with the isomorphisms

$$BK/K \simeq B/B \cap K, \quad BK'/K' \simeq B/B \cap K'$$

implies

$$B \cap K / B \cap K' \simeq K/K'.$$

By definition, we have isomorphisms $\Lambda_{G/K} \simeq \mathcal{X}(B)^{B \cap K}$ and $\Lambda_{G/K'} \simeq \mathcal{X}(B)^{B \cap K'}$. Moreover, the restriction gives an homomorphism

$$\mathcal{X}(B)^{B \cap K'} \longrightarrow \mathcal{X}(B \cap K)^{B \cap K'} = \mathcal{X}(B \cap K / B \cap K')$$

which is surjective by following lemma and whose kernel is $\mathcal{X}(B)^{B \cap K}$. Therefore we get

$$\Lambda_{G/K'} / \Lambda_{G/K} \simeq \mathcal{X}(B \cap K / B \cap K') \simeq \mathcal{X}(K/K').$$

\square

Lemma 3.5. *Let $H \subset B$ be a subgroup. Then the restriction of characters $\mathcal{X}(B) \rightarrow \mathcal{X}(H)$ is surjective.*

Proof. Let U be the unipotent radical of B , then $\mathcal{X}(B) = \mathcal{X}(B/U)$ and $\mathcal{X}(H) = \mathcal{X}(H/H \cap U)$. Since B/U is a torus and $H/H \cap U$ is a (possibly not connected) subtorus, the restriction $\mathcal{X}(B/U) \rightarrow \mathcal{X}(H/H \cap U)$ is surjective. \square

Coming back to our situation, since K/K' is a finite diagonalizable group, it is isomorphic to its character group. Thus we obtain

Corollary 3.6. *Let $G/K \simeq Z \subset X$ be an orbit and let $G/K' \simeq Z' = p^{-1}(Z) \subset \tilde{X}$ be the corresponding orbit, with $K' \subset K$; then*

$$\Lambda_{Z'}/\Lambda_Z \simeq K/K'.$$

Let $y = [v^-] \in Y^{B^-}$ be the unique fixed point by B^- (where $v^- \in V$ is a lowest weight vector) and let $\eta \in (V^*)^{(B)}$ be the maximal vector defined by $\langle \eta, v^- \rangle = 1$: then $\langle \eta, v_0 \rangle \neq 0$, otherwise it would be $\eta|_X = 0$, which is absurd since $y \in X$.

Consider the affine open subset $X_0 = X \cap \mathbf{P}(V)_\eta$ defined by the non-vanishing of η : it is P -stable and, since it intersects the closed orbit, it intersects every orbit of X . Set L the Levi subgroup defined by $L := P \cap \text{Stab}(y)$ and recall that there exists an affine closed L -stable subvariety $S_X \subset X_0$ containing y as a fixed point and containing an open $(B \cap L)$ -orbit such that the multiplication morphism

$$\begin{array}{ccc} R_u(P) \times S_X & \longrightarrow & X_0 \\ (g, s) & \longmapsto & gs, \end{array}$$

is a P -equivariant isomorphism ([B2], Theorem 1.4). Since $k[S_X//L] = k[S_X]^L = k$, we get that S_X possesses a unique closed L -orbit, namely the B^- -fixed point y .

Lemma 3.7. *Let $D \in \Delta(G/H)$; then $\overline{D} \supset Y$ if and only if $\eta|_D \neq 0$.*

Proof. Suppose $D \in \Delta(G/H)$ is such that $\eta|_D \neq 0$, then $\overline{D} \cap \mathbf{P}(V)_\eta \subset \mathbf{P}(V)_\eta$ is non-empty, closed and P -stable. Therefore $\overline{D} \cap S_X$ is non-empty, L -stable and closed in S_X , hence $y \in \overline{D} \cap S_X$; this implies that $Y_B = By \subset \overline{D}$, that is $\overline{D} \supset Y$.

Suppose conversely that $D \in \Delta(G/H)$ is such that $D \subset \mathbf{P}(\ker(\eta))$: then we obtain $\overline{D} \subset \mathbf{P}(\ker(\eta))$, which implies $\overline{D} \not\supset Y$. \square

Since X_0 intersects every orbit of X , by previous lemma we obtain the equality

$$X_0 = X \setminus \bigcup_{D \in \Delta(G/H): \overline{D} \not\supset Y} \overline{D}.$$

Let $Z \subset X$ be an orbit; then we obtain a commutative diagram

$$\begin{array}{ccc} \overline{Z'} & \xrightarrow{\quad} & \tilde{X} \\ \downarrow & & \downarrow \\ \overline{Z} & \xrightarrow{\quad} & X \end{array}$$

If $Z_0 = \overline{Z} \cap \mathbf{P}(V)_\eta$, then, as for X_0 , we obtain the equality

$$Z_0 = \overline{Z} \setminus \bigcup_{D \in \Delta(Z): \overline{D} \not\supset Y} \overline{D}.$$

Let $\tilde{X}_0 = p^{-1}(X_0) \subset \tilde{X}$ and $Z'_0 = p^{-1}(Z_0) \subset \overline{Z'}$ be respectively the inverse images of X_0 and Z_0 in \tilde{X} . Then

$$Z'_0 = \tilde{X}_0 \cap \overline{Z'}, \quad Z_0 = X_0 \cap \overline{Z}.$$

Previous diagram is preserved by restriction to such affine open sets (where horizontal arrows are closed immersions):

$$\begin{array}{ccc} Z'_0 & \hookrightarrow & \tilde{X}_0 \\ \downarrow & & \downarrow \\ Z_0 & \hookrightarrow & X_0; \end{array}$$

finally, considering the respective rings of functions, we obtain

$$\begin{array}{ccc} k[X_0] & \hookrightarrow & k[\tilde{X}_0] \\ \downarrow & & \downarrow \\ k[Z_0] & \hookrightarrow & k[Z'_0]. \end{array}$$

Proposition 3.8 ([K1], Theorem 1.3). *Every B -semiinvariant function $f \in k[Z_0]^{(B)}$ (resp. $f \in k[Z'_0]^{(B)}$) can be extended to a B -semiinvariant function $f' \in k[X_0]^{(B)}$ (resp. $f' \in k[\tilde{X}_0]^{(B)}$).*

Corollary 3.9. *The lattices Λ_Z and $\Lambda_{Z'}$ are canonically identified with the following sublattices of $\Lambda_{G/H}$ (where χ_f denotes the weight of $f \in k(G/H)^{(B)}$):*

$$\Lambda_Z = \left\{ \chi_f \in \Lambda_{G/H} : f \in k(X)^{(B)} \text{ is such that } f|_Z \text{ and } f^{-1}|_Z \text{ exist in } k(Z) \right\}$$

$$\Lambda_{Z'} = \left\{ \chi_f \in \Lambda_{G/H} : f \in k(\tilde{X})^{(B)} \text{ is such that } f|_{Z'} \text{ and } f^{-1}|_{Z'} \text{ exist in } k(Z') \right\}.$$

Proof. Up to a scalar factor, each B -semiinvariant function is univocally determined by its weight and, by Proposition 3.8, restriction gives isomorphisms of semigroups

$$\left\{ f \in k[X_0]^{(B)} : f|_Z \neq 0 \right\} \xrightarrow{\sim} k[Z_0]^{(B)}$$

$$\left\{ f \in k[\tilde{X}_0]^{(B)} : f|_{Z'} \neq 0 \right\} \xrightarrow{\sim} k[Z'_0]^{(B)}.$$

On the other hand, each B -semiinvariant rational function on X (resp. on \tilde{X} , \overline{Z} , $\overline{Z'}$) can be written as a quotient of two B -semiinvariant regular functions on X_0 (resp. on \tilde{X}_0 , Z_0 , Z'_0). Thus, considering the lattices generated by such semigroups, we obtain the claimed identification. \square

Definition 3.10. If Λ is a finitely generated free \mathbf{Z} -module and $\Gamma \subset \Lambda$ is a submodule, then the *saturation* of Γ in Λ is the submodule $\overline{\Gamma} = \Gamma_{\mathbf{Q}} \cap \Lambda$. If $\Gamma = \overline{\Gamma}$, then Γ is said to be *saturated* in Λ .

Proposition 3.11. *Fix an orbit $Z \subset X$ and set $Z' \subset \tilde{X}$ its corresponding orbit.*

- (i) $\Lambda_{Z'}$ is the saturation of Λ_Z in $\Lambda_{G/H}$.
- (ii) If $Z \simeq Z'$, then $\overline{Z'} \subset \tilde{X}$ is the normalization of $\overline{Z} \subset X$.

Proof. Since $\overline{Z'} \subset \tilde{X}$ is normal (see [B2], Corollary 2.3.1), it follows that $\Lambda_{Z'} \subset \Lambda_{G/H}$ is a saturated sublattice. Since $[\Lambda_{Z'} : \Lambda_Z] = [K : K'] < \infty$, then we get that $\Lambda_{Z'} = \Lambda_{G/H} \cap (\Lambda_Z)_{\mathbf{Q}}$ is the saturation of Λ_Z in $\Lambda_{G/H}$, thus we obtain (a).

As for (b), set $p' : \tilde{Z} \rightarrow \overline{Z}$ the normalization of \overline{Z} ; then we get a commutative diagram

$$\begin{array}{ccc} \overline{Z'} & \xrightarrow{\tilde{p}} & \tilde{Z} \\ & \searrow p & \downarrow p' \\ & & \overline{Z} \end{array}$$

Since p and p' are both finite morphisms, \tilde{p} too is a finite morphism; on the other hand, by Zariski main theorem the fibers of \tilde{p} are connected, therefore \tilde{p} is bijective. Applying Zariski main theorem once more, we get that \tilde{p} is an isomorphism. \square

If $D \in \Delta(G/H)$, denote \overline{D} and \tilde{D} its closure respectively in X and in \tilde{X} .

Lemma 3.12. *Let $D \in \Delta(G/H)$; if $Z \subset X$ is an orbit and $Z' = p^{-1}(Z) \subset \tilde{X}$ is the corresponding orbit, then $\overline{D} \supset Z$ if and only if $\tilde{D} \supset Z'$.*

Proof. Observe first of all that $\overline{D} = p(\tilde{D})$: indeed, on one hand $p(\tilde{D}) \supset \overline{D}$ because p is a closed map, while on the other hand $p^{-1}(\overline{D}) \supset \tilde{D}$, which implies $\overline{D} \supset p(\tilde{D})$.

Set $Z_B \subset Z$ the open B -orbit; then by Lemma 3.4 $p^{-1}(Z_B) = Z'_B$ is the open B -orbit of Z' . Suppose that $\overline{D} \supset Z$ and fix $z_0 \in Z_B$; if $z'_0 \in p^{-1}(z_0) \cap \tilde{D}$, then, since \tilde{D} is B -stable, we obtain $Z'_B = Bz'_0 \subset \tilde{D}$, which implies $Z' \subset \tilde{D}$. Suppose on the contrary that $\tilde{D} \not\supset Z'$: then $\overline{D} = p(\tilde{D}) \supset p(Z') = Z$. \square

Combining previous lemma together with Lemma 3.7, we obtain that the set of colors $\Delta_Y(\tilde{X}_\delta) \subset \Delta(G/H)$ whose closure in \tilde{X} contains the closed orbit Y is

$$\Delta_Y(\tilde{X}_\delta) = \{D \in \Delta(G/H) : \eta|_D \neq 0\}.$$

Recall that any normal embedding of a spherical homogeneous space $G/H \hookrightarrow X$ which possesses a unique closed orbit Y is completely determined by its *colored cone*: this is the couple $(\mathcal{C}_Y(X), \Delta_Y(X))$, where $\mathcal{C}_Y(X) \subset (\Lambda_{G/H}^\vee)_{\mathbf{Q}}$ is the cone generated by the homomorphisms induced by the rational discrete valuations associated to B -stable (possibly G -stable) prime divisors of X which contain Y (see [K1]). Since \tilde{X}_δ is simple and complete, the support of its colored cone contains the G -invariant valuation cone $\mathcal{V}_{G/H}$. Therefore the colored cone of \tilde{X}_δ is given by the couple

$$(\mathcal{C}_Y(\tilde{X}_\delta), \Delta_Y(\tilde{X}_\delta)),$$

where $\mathcal{C}_Y(\tilde{X}_\delta) \subset (\Lambda_{G/H}^\vee)_{\mathbf{Q}}$ is the cone generated by the G -invariant valuation cone $\mathcal{V}_{G/H}$ together with $\rho_{G/H}(\Delta_Y(\tilde{X}_\delta))$.

4. FAITHFUL DIVISORS.

Let M be a wonderful variety and fix a base point x_0 in the open orbit so that $Gx_0 \simeq G/H$, where H is the stabilizer of x_0 . Set $\mathcal{S} = (\Sigma, S^p, \mathbf{A})$ the spherical system of M and set $\Delta = \Delta(G/H)$ the set of colors of M .

A subset $\Delta^* \subset \Delta$ is said to be *distinguished* if there exists $\delta \in \mathbf{N}_{>0}\Delta^*$ such that $\langle \rho_{G/H}(\delta), \gamma \rangle \geq 0$, for every $\gamma \in \Sigma$. If $H' \supset H$ is a sober subgroup such that H'/H is connected and if $\phi : G/H \rightarrow G/H'$ is the projection, then the subset of colors

$$\Delta_\phi = \{D \in \Delta : \overline{\phi(D)} = G/H'\}$$

is distinguished; conversely, if $\Delta^* \subset \Delta$ is a distinguished subset, then there exists a unique sober subgroup $H' \supset H$ with H'/H connected such that $\Delta^* = \Delta_\phi$, where $\phi : G/H \rightarrow G/H'$ is the projection. This is the content of following theorem:

Theorem 4.1 ([K1], Theorem 4.4; [L2], Lemma 3.3.1). *There is a bijection as follows*

$$\{ \Delta^* \subset \Delta \text{ distinguished subset} \} \longleftrightarrow \left\{ \begin{array}{l} H' \subset G \text{ sober :} \\ H \subset H' \text{ and } H'/H \text{ connected} \end{array} \right\}$$

Moreover, if $H' \supset H$ is a sober subgroup such that H'/H is connected and if $\Delta^* \subset \Delta$ is the corresponding distinguished subset, then the projection $G/H \rightarrow G/H'$ identifies the set of colors of G/H' with $\Delta \setminus \Delta^*$. In particular, if $G/H' \hookrightarrow M'$ is

the canonical embedding, then $\text{Pic}(M')$ is identified with the sublattice $\mathbf{Z}[\Delta \setminus \Delta^*] \subset \mathbf{Z}\Delta = \text{Pic}(M)$.

Remark 4.2. If $H \subset H'$ and Δ^* are as in previous theorem and if $\phi : M \rightarrow M'$ is the map which extends the natural projection, consider the immersion $\Lambda_{G/H'} \hookrightarrow \Lambda_{G/H}$ and set $N(\Delta^*) := \Lambda_{G/H'}^\perp \subset (\Lambda_{G/H}^\vee)_{\mathbf{Q}}$: it is a linear subspace which contains $\rho_{G/H}(\Delta^*)$. Then the lattice $\Lambda_{G/H'}$ is identified with a sublattice of $\Lambda_{G/H}$ as follows:

$$\Lambda_{G/H'} = \Lambda_{G/H} \cap N(\Delta^*)^\perp;$$

therefore it is saturated as a sublattice of $\Lambda_{G/H}$ (see [K1], Lemma 4.3).

Moreover, if Δ' is the set of colors of M' and if $M'_0 = M' \setminus \bigcup_{\Delta'} D$, then $\phi^{-1}(M'_0) = M \setminus \bigcup_{\Delta \setminus \Delta^*} D$ and, since the fibers of ϕ are complete and connected, we get $k[M'_0] = k[\phi^{-1}(M'_0)]$. Looking at B -semiinvariant functions, we get then the identification of semigroups

$$k[M'_0]^{(B)}/_{k^*} \simeq \{ \sigma \in -\mathbf{N}\Sigma : \langle \rho_{G/H}(D), \sigma \rangle = 0, \forall D \in \Delta^* \} =: -\mathbf{N}\Sigma/\Delta^*.$$

If M' is smooth, then such semigroup is free; conversely it is known (even if there is no proof in the literature) that, given any distinguished subset $\Delta^* \subset \Delta$, the semigroup $\mathbf{N}\Sigma/\Delta^*$ is free, i. e. that M' is necessarily smooth. Assuming such property, then the spherical system of M' is the *quotient spherical system* $\mathcal{S}/\Delta^* = (\Sigma/\Delta^*, S^p/\Delta^*, \mathbf{A}/\Delta^*)$, where

- Σ/Δ^* is the set of indecomposable elements of the semigroup $\mathbf{N}\Sigma/\Delta^*$;
- $S^p/\Delta^* = S^p \cup \{ \alpha \in S : \Delta(\alpha) \subset \Delta^* \}$;
- $\mathbf{A}/\Delta^* = \bigcup_{\alpha \in S \cap \Sigma/\Delta^*} \mathbf{A}(\alpha)$, and the pairing is obtained by restriction.

Corollary 4.3. Let H be a wonderful subgroup, let $K \supset H$ be any sober subgroup and let $\phi : G/H \rightarrow G/K$ the projection; set Δ the set of colors of G/H .

- (i) The subset $\Delta_\phi \subset \Delta$ of colors which map dominantly on G/K is distinguished.
- (ii) K and H have the same dimension if and only if $\Delta_\phi = \emptyset$.

Proof. Set K^0 the identity component of K and set $K^* := HK^0 \subset K$. Since $H \subset N_G(K) = N_G(K^0)$, K^* is a sober subgroup of G . Set $\phi' : G/H \rightarrow G/K^*$ the projection. Since $\dim K^* = \dim K$, we get $\Delta_{\phi'} = \Delta_\phi$; on the other hand, since

$$K^*/H \simeq K^0/K^0 \cap H$$

is connected, $\Delta_{\phi'}$ is distinguished by Theorem 4.1 and we obtain (a), while (b) follows straightforward. \square

If $G/P_M \simeq Y_M \subset M$ is the closed orbit and if $\omega : \text{Pic}(M) \rightarrow \text{Pic}(Y_M) \simeq \mathcal{X}(P_M) \subset \mathcal{X}(B)$ and $\psi : \text{Pic}(M) \rightarrow \text{Pic}(G/H) \simeq \mathcal{X}(H)$ are the restrictions of linear bundles to closed orbit and to open orbit respectively, then we get a commutative diagram

$$\begin{array}{ccc} \text{Pic}(M) & \xrightarrow{\psi} & \mathcal{X}(H) \\ \omega \downarrow & & \downarrow \\ \mathcal{X}(B) & \longrightarrow & \mathcal{X}(B \cap H) \end{array}$$

which identifies $\text{Pic}(M)$ with the fibred product

$$\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H) = \{ (\lambda, \chi) \in \mathcal{X}(B) \times \mathcal{X}(H) : \lambda|_{B \cap H} = \chi|_{B \cap H} \}.$$

Under this identification, a divisor δ is generated by its global sections if and only if the weight $\omega(\delta)$ is dominant. The map ω is combinatorially described as follows ([Fo], Theorem 2.2):

$$\omega(D) = \begin{cases} \sum_{D \in \Delta(\alpha)} \omega_\alpha & \text{if } D \in \mathbf{A} \cup \Delta^b \\ 2\omega_\alpha & \text{if } D \in \Delta(\alpha) \subset \Delta^{2a} \end{cases},$$

where ω_α is the fundamental dominant weight associated to $\alpha \in S$.

Let δ be a divisor generated by its global sections on M : this means that, up to linear equivalence, we can write $\delta = \sum_{\Delta} n(\delta, D)D$ with $n(\delta, D) \geq 0$ for every $D \in \Delta$. If $s \in \Gamma(M, \mathcal{O}(\delta))$ is the canonical section, then we can identify the submodule $\langle Gs \rangle \subset \Gamma(M, \mathcal{O}(\delta))$ generated by s with the simple G -module $V_{\omega(\delta)}^*$. Thus we get a projective morphism

$$\phi_\delta : M \longrightarrow \mathbf{P}(V_{\omega(\delta)}^*).$$

Given any simple G -module V , the H -eigenspace $V_\chi^{(H)}$ where H acts by a fixed character $\chi \in \mathcal{X}(H)$ has dimension at most one: this is a well known property of spherical subgroups, which is in fact equivalent to the sphericity of H . Since the line $\phi_\delta(x_0)$ is fixed by H , set $\chi_\delta \in \mathcal{X}(H)$ the character by which H acts on it: then χ_δ coincides with the image of δ by the restriction $\psi : \text{Pic}(M) \rightarrow \mathcal{X}(H)$. In this way the fiber of $\omega : \text{Pic}(M) \rightarrow \mathcal{X}(B)$ over a weight λ is naturally indexed by H -fixed points in $\mathbf{P}(V_\lambda^*)$ and is identified with the set

$$\{\chi \in \mathcal{X}(H) : (V_\lambda^*)_\chi^{(H)} \neq 0\}.$$

In general, the stabilizer of a point in $\mathbf{P}(V_{\omega(\delta)}^*)^H$ will strictly contain H ; however it is known to be wonderful:

Theorem 4.4 ([BL], Corollary 2.4.2). *A spherical subgroup K is spherically closed if and only if it occurs as the stabilizer of a point in some simple projective space.*

Set $\text{Supp}_\Delta(\delta)$ the support of δ on Δ , defined as follows:

$$\text{Supp}_\Delta(\delta) = \{D \in \Delta : n(\delta, D) \neq 0\}.$$

As a consequence of Theorem 4.1, we get the following corollary.

Corollary 4.5. *Let M be a wonderful variety and let $\delta \in \mathbf{N}\Delta$ be a divisor generated by its global sections; consider the associated morphism $\phi_\delta : M \rightarrow \mathbf{P}(V_{\omega(\delta)}^*)$. Fix $v_0 \in (V_{\omega(\delta)}^*)_\chi^{(H)}$ a representative of the line $\phi_\delta(x_0)$, where $\chi = \psi(\delta)$, and set $K = \text{Stab}[v_0] \subset G$ its stabilizer. Then the correspondence of Theorem 4.1 gives a bijection*

$$\left\{ \begin{array}{l} \Delta^* \subset \Delta \text{ distinguished subset} : \\ \Delta^* \cap \text{Supp}_\Delta(\delta) = \emptyset \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} H' \subset G \text{ sober} : \\ H \subset H' \subset K \text{ and} \\ H'/H \text{ connected} \end{array} \right\}$$

Proof. Let $H' \supset H$ be a sober subgroup such that H'/H is connected and set $\Delta^* \subset \Delta$ the distinguished subset of colors which corresponds to it; set $M' = M(G/H')$ its canonical embedding. Then the projection $G/H \rightarrow G/H'$ extends to a morphism $M \rightarrow M'$ and pullback identifies $\text{Pic}(M')$ with the submodule of $\text{Pic}(M) = \mathbf{Z}\Delta$ generated by $\Delta \setminus \Delta^*$. On the other hand, $H' \subset K$ if and only if χ extends to a character of H' which coincides with $\omega(\delta)$ on $B \cap H'$: this is equivalent to $\delta \in \text{Pic}(M') = \mathbf{Z}[\Delta \setminus \Delta^*]$, i. e. $\text{Supp}_\Delta(\delta) \subset \Delta \setminus \Delta^*$. \square

Definition 4.6 ([BL], §2.4.3). Let M be a spherically closed wonderful variety and set Δ its set of colors. A divisor $\delta \in \mathbf{N}\Delta$ generated by its global sections is said to be *faithful* if it satisfies the following conditions:

- (1) Every non-empty distinguished subset of Δ intersects $\text{Supp}_\Delta(\delta)$;

(2) If $\alpha \in \Sigma_\ell(M)$ is a loose spherical root, then $n(\delta, D_\alpha^+) \neq n(\delta, D_\alpha^-)$.

Theorem 4.7 ([BL], Proposition 2.4.3). *Let δ be a divisor generated by its global sections on a spherically closed wonderful variety M , with base point x_0 and with open orbit $G/H \simeq Gx_0$. Then the associated morphism $\phi_\delta : M \rightarrow \mathbf{P}(V_{\omega(\delta)}^*)$ restricts to an embedding $G/H \hookrightarrow \mathbf{P}(V_{\omega(\delta)}^*)$ if and only if δ is faithful.*

Proof. Set $V = V_{\omega(\delta)}^*$ and let $v_0 \in V_\chi^{(H)}$ be a representative of the line $\phi_\delta(x_0)$, where $\chi \in \mathcal{X}(H)$ is defined by $\chi = \psi(\delta)$; let Δ be the set of colors of M .

Suppose that ϕ_δ restricts to an embedding $G/H \hookrightarrow \mathbf{P}(V)$. Then by previous lemma we obtain (1). Suppose ab absurdo that (2) fails and let $\alpha \in \Sigma_\ell(M) \subset S \cap \Sigma$ be a loose spherical root such that $n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)$. If $\gamma(\alpha) \in \text{Aut}_G(M) = N_G(H)/H$ is the corresponding automorphism which fixes pointwise the prime G -stable divisor associated to α , then $\gamma(\alpha)$ exchanges D_α^+ and D_α^- and fixes every other color $D \in \Delta \setminus \Delta(\alpha)$; therefore $\gamma(\alpha)$ fixes δ . The action of $\text{Aut}_G(M)$ on $\text{Pic}(M) = \mathbf{Z}\Delta \simeq \mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H)$ is defined extending by linearity the right action of $N_G(H)/H$ on Δ , i. e. letting act $N_G(H)$ on $\mathcal{X}(H)$. Therefore, if $g \in N_G(H)$ is a representative of $\gamma(\alpha)$, then $\chi^g = \chi$, i. e. g moves the line $[v_0]$, acted on by H by the character χ , in a line where H acts by the same character $\chi = \chi^g$: since H is spherical, such a line is unique, i. e. $g \in \text{Stab}[v_0] = H$, which is absurd.

Suppose viceversa that δ is a faithful divisor. By (1) together with Corollary 4.3 and Corollary 4.5, we obtain $\dim H = \dim \text{Stab}[v_0]$; therefore, by Theorem 3.2, $H \subset \text{Stab}[v_0] \subset N_G(H)$. Suppose ab absurdo that there exists $g \in \text{Stab}[v_0] \setminus H$. Then $\chi^g = \chi$, so the corresponding G -automorphism of M fixes δ : therefore by (2) we obtain that every color $D \in \text{Supp}_\Delta(\delta)$ is fixed by g . On the other hand, since H is spherically closed, every element in $N_G(H) \setminus H$ acts non-trivially on Δ . If $\alpha \in S$ is such that $D \in \Delta(\alpha)$ is moved by g , then we get $\alpha \in \Sigma_\ell(M) \subset S \cap \Sigma$ and $\Delta(\alpha) = \{D, D \cdot g\}$: therefore $n(\delta, D) = n(\delta, D \cdot g) = 0$, which is absurd by (2). \square

Corollary 4.8. *In the same hypotheses of Corollary 4.5, suppose moreover that every distinguished subset of Δ intersects $\text{Supp}_\Delta(\delta)$; set*

$$\Sigma(\delta) = \{\alpha \in \Sigma_\ell(M) : \alpha \notin S \text{ or } n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)\}.$$

- (i) $H = \text{Stab}[v_0]$ if and only if $\Sigma(\delta) = \emptyset$.
- (ii) The spherical system of $\text{Stab}[v_0]$ is $\mathcal{S}' = (\Sigma', S^p, \mathbf{A}')$, where

$$\Sigma' = (\Sigma \setminus \Sigma(\delta)) \cup 2\Sigma(\delta) \quad \text{and} \quad \mathbf{A}' = \bigcup_{\alpha \in S \cap \Sigma'} \mathbf{A}(\alpha).$$

Proof. Since (a) follows immediately by previous theorem, we only need to show (b). For every $\sigma \in \Sigma_\ell(\mathcal{S})$, the quotient $M/\gamma(\sigma)$ is easily proved to be a wonderful variety, whose spherical system is $\mathcal{S}^* = (\Sigma^*, S^p, \mathbf{A}^*)$, where $\Sigma^* = (\Sigma \setminus \{\sigma\}) \cup \{2\sigma\}$ and where $\mathbf{A}^* = \bigcup_{\alpha \in S \cap \Sigma^*} \mathbf{A}(\alpha)$. If $g \in N_G(H)$ is a representative of the coset corresponding to $\gamma(\sigma)$, then $M/\gamma(\sigma) = M(G/H_\sigma)$, where H_σ is the subgroup generated by H together with g . The first part of the proof of previous theorem then shows that H_σ fixes $[v_0]$; therefore we obtain a commutative diagram

$$\begin{array}{ccc} M & & \\ \downarrow & \searrow \phi_\delta & \\ M/\gamma(\sigma) & \xrightarrow{\phi_{\delta_\sigma}} & \mathbf{P}(V_{\omega(\delta)}^*) \end{array}$$

where δ_σ is the pullback of $\mathcal{O}(1)$ on $M/\gamma(\sigma)$. Consider now the quotient M/Γ_δ , where $\Gamma_\delta \subset \text{Aut}_G(X)$ is the subgroup generated by the elements $\gamma(\sigma)$, with $\sigma \in$

$\Sigma(\delta)$: then, by previous discussion and by Theorem 4.7, it follows that M/Γ_δ is a spherically closed wonderful variety endowed with a divisor δ' such that

$$\phi_{\delta'} : M/\Gamma_\delta \longrightarrow \mathbf{P}(V_{\omega(\delta)}^*)$$

restricts to an embedding of the open orbit. \square

Remark 4.9. In previous corollary's hypotheses, the assumption that every distinguished subset of colors intersects $\text{Supp}_\Delta(\delta)$ (which is equivalent to assume that $\dim H = \dim \text{Stab}[v_0]$) actually involves no loss of generality: we can always reduce to this case considering, instead of M , the wonderful variety whose generic stabilizer is the spherical closure of the sober subgroup corresponding to the maximal distinguished subset of colors $\Delta(\delta) \subset \Delta$ which does not intersect $\text{Supp}_\Delta(\delta)$.

5. ORBITS IN X_δ AND IN \tilde{X}_δ .

Let V be a simple G -module and suppose $G/H \hookrightarrow \mathbf{P}(V)$ is a spherical orbit; then, by Theorem 4.4, H is spherically closed. Let M be its wonderful completion; set $\mathcal{S} = (\Sigma, S^p, \mathbf{A})$ its spherical system and set Δ the set of colors. Consider the morphism $\phi : M \rightarrow \mathbf{P}(V)$ and set $\delta = \phi^* \mathcal{O}(1) \in \text{Pic}(M)$ the pullback of the hyperplane bundle: by construction, $\phi = \phi_\delta$ is the projective morphism associated to δ , which is a faithful divisor on M . Set $X_\delta = \phi_\delta(M) \subset \mathbf{P}(V)$ and set $p : \tilde{X}_\delta \rightarrow X_\delta$ the normalization; set Y the closed orbit in X_δ (which is identified with the closed orbit of \tilde{X}_δ). Then we get a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\phi}_\delta} & \tilde{X}_\delta \\ & \searrow \phi_\delta & \downarrow p \\ & & X_\delta \subset \mathbf{P}(V) \end{array}$$

and $\Delta_Y(X_\delta) = \Delta_Y(\tilde{X}_\delta)$ is canonically identified with $\Delta \setminus \text{Supp}_\Delta(\delta)$.

If M is a strict wonderful variety and if $\delta \in \text{Pic}(M)$ is a faithful divisor, then the variety X_δ depends only on the support of δ ([BGM], Corollary 3.7). As shown by following Corollary 7.4, this is not true if M is not strict.

Let $G/K \simeq Z \subset X_\delta$ be an orbit and let $G/K' \simeq Z' = p^{-1}(Z)$ be the corresponding orbit in \tilde{X}_δ ; let $G/K_W \simeq W \subset M$ be an orbit which maps on Z and choose the stabilizers so that $K_W \subset K' \subset K$. Therefore we obtain inclusions

$$\Lambda_Z \subset \Lambda_{Z'} \subset \Lambda_W \subset \Lambda_{G/H};$$

since Λ_W is saturated in $\Lambda_{G/H}$, as in Proposition 3.11 we obtain that $\Lambda_{Z'}$ is the saturation of Λ_Z in Λ_W . Consider the diagram

$$\begin{array}{ccc} \overline{W} & \xrightarrow{\tilde{\phi}_\delta} & \overline{Z'} \\ & \searrow \phi_\delta & \downarrow p \\ & & \overline{Z} \subset \mathbf{P}(V) \end{array}$$

Let $\delta_W \in \text{Pic}(\overline{W})$ be the pullback of δ on \overline{W} ; then the restriction of ϕ_δ to \overline{W} equals the map ϕ_{δ_W} associated to δ_W .

Proposition 5.1. *Let K be the stabilizer of an orbit $Z \subset X_\delta$ and let $K_W \subset K$ be the stabilizer of an orbit $W \subset M$ mapping on Z ; let K' be the stabilizer of the corresponding orbit $Z' = p^{-1}(Z) \subset \tilde{X}_\delta$. Then $K' = K_W K^0$ is the maximal sober subgroup such that $K_W \subset K' \subset K$ and K'/K_W is connected.*

Proof. Set $\Delta(\delta_W) \subset \Delta(W)$ the maximal distinguished subset of colors of \overline{W} which does not intersect the support of δ_W and consider the sober subgroup $K^* = K_W K^0$: by Corollary 4.5, K^* is the maximal sober subgroup of G containing K_W such that $K^* \subset K$ and K^*/K_W is connected. Since $K_W \subset K'$ and since $K^0 = (K')^0$, we get the inclusion $K^* \subset K'$; since K^* is normal in K' , by Lemma 3.4 we get that $K^* = K'$ if and only if $\Lambda_{G/K^*} = \Lambda_{Z'}$.

Since \overline{W} is a localization of M , Λ_W is a saturated sublattice of $\Lambda_{G/H}$; on the other hand, by Remark 4.2 it follows that Λ_{G/K^*} also is saturated in Λ_W , therefore Λ_{G/K^*} is saturated in $\Lambda_{G/H}$ too. Since $[\Lambda_{G/K^*} : \Lambda_Z] = [K : K^*] < \infty$, we get that Λ_{G/K^*} is the saturation of Λ_Z in $\Lambda_{G/H}$; therefore, by Proposition 3.11, Λ_{G/K^*} equals $\Lambda_{Z'}$ and we get the equality $K^* = K'$. \square

Corollary 5.2. *In the same notations of previous lemma, the following conditions are equivalent:*

- (i) Z and Z' are isomorphic;
- (ii) K/K_W is connected.

If moreover M is strict, then K is the spherical closure of K' .

Fix now an orbit $G/K \simeq Z \subset X_\delta$ and let $\Sigma_Z \subset \mathbf{Z}\Sigma$ be the set of spherical roots of its wonderful completion. If $\gamma \in \Sigma_Z$ is defined by $\gamma = \sum_{\sigma \in \Sigma} n(\gamma, \sigma)\sigma$, set

$$\text{Supp}_\Sigma(\gamma) = \{\sigma \in \Sigma : n(\gamma, \sigma) \neq 0\}$$

its support over Σ . Set

$$\Sigma(Z) = \bigcup_{\gamma \in \Sigma_Z} \text{Supp}_\Sigma(\gamma) \subset \Sigma,$$

and define W_Z the corresponding orbit in M : following proposition shows that it is the minimal orbit in M which maps on Z .

Proposition 5.3. *Let $Z \subset X_\delta$ be an orbit and let $W_Z \subset M$ the orbit defined by $\Sigma(Z) \subset \Sigma$. Then W_Z maps on Z and every other orbit which maps on Z contains W_Z in its closure.*

Proof. Let $W \subset M$ be an orbit mapping on Z and let $\Sigma_W \subset \Sigma$ the relative set of spherical roots. Since $\phi_\delta(W) = Z$, we get $\Sigma_Z \subset \mathbf{Z}\Sigma_W$, i. e. $\Sigma(Z) \subset \Sigma_W$: this shows the inclusion $W_Z \subset \overline{W}$.

In order to prove that W_Z maps on Z , consider the commutative diagram

$$\begin{array}{ccccc} \overline{W_Z} & \xrightarrow{\quad} & \overline{W} & & \\ \phi'_\delta \downarrow & & \phi'_\delta \downarrow & \searrow \phi_\delta & \\ \phi'_\delta(\overline{W_Z}) & \xrightarrow{\quad} & M(Z) & \longrightarrow & \overline{Z} \end{array}$$

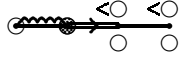
where $\phi'_\delta : \overline{W} \rightarrow M(Z)$ is the map induced by the restriction $\phi_\delta : \overline{W} \rightarrow \overline{Z}$, which factors through $M(Z)$, the wonderful completion of Z . Then the rank of the wonderful subvariety $\phi'_\delta(\overline{W_Z}) \subset M(Z)$ is the rank of the lattice $\Lambda_Z \cap \Lambda_{W_Z}$, which by construction coincides with Λ_Z . This shows that $\phi'_\delta(\overline{W_Z})$ and $M(Z)$ have the same rank, which implies the equality $\phi'_\delta(\overline{W_Z}) = M(Z)$, i. e. $\phi_\delta(W_Z) = Z$. \square

Corollary 5.4. *Different orbits in X_δ have different sets of spherical roots; in particular different orbits in X_δ are never isomorphic.*

Proof. It follows immediately by previous proposition since, if $Z \subset X_\delta$ is an orbit, the set $\Sigma(Z) \subset \Sigma$ is completely determined by Σ_Z . \square

Remark 5.5. If M is strict and if $Y_M \subset M$ is the closed orbit, then the restriction $\omega : \text{Pic}(M) \rightarrow \text{Pic}(Y_M)$ is injective ([Pe], Lemma 14); equivalently, this means that a strict wonderful subgroup never fixes two different lines in the same simple G -module. Therefore previous corollary's claim is obvious under the hypothesis of strictness. However, if H is not strict, it could fix more than one line in the same simple G -module. Previous corollary shows that, if a spherical subgroup fixes two different lines $[v]$ and $[w]$ in the same simple G -module V , then there is no spherical orbit in $\mathbf{P}(V)$ containing both $[v]$ and $[w]$ in its closure. For instance, this occurs in the following example.

Example 5.6. Consider the wonderful variety whose spherical system is expressed by following spherical diagram



Following [BL] §3.6, this spherical system is geometrically realizable and corresponds to a spherically closed wonderful variety M ; set H its generic stabilizer. Set $\delta_1 = D_{\alpha_2} + D_{\alpha_4}^+$ and $\delta_2 = D_{\alpha_2} + D_{\alpha_4}^-$; then δ_1 and δ_2 are both faithful divisors on M and $\omega(\delta_1) = \omega(\delta_2) = \omega_{\alpha_2} + \omega_{\alpha_4}$. Moreover, since H is self-normalizing, the open orbits in X_{δ_1} and in X_{δ_2} are not the same one: therefore the projective space $\mathbf{P}(V_{\omega_{\alpha_2} + \omega_{\alpha_4}}^*)$ contains two different orbits both isomorphic to G/H .

By Proposition 5.1 together with Corollary 5.4 we obtain the following combinatorial criterion to determine whether or not two orbits of M map to the same orbit of X_δ :

Corollary 5.7. *Two orbits $W_1, W_2 \subset M$ map to the same orbit in X_δ if and only if*

$$\Sigma_{W_1} / \Delta(\delta_{W_1}) = \Sigma_{W_2} / \Delta(\delta_{W_2}),$$

where δ_{W_i} is the pullback of δ to $\overline{W_i}$ and where $\Delta(\delta_{W_i})$ is the maximal distinguished subset of colors of W_i not intersecting the support of δ_{W_i} .

Remark 5.8. Unlike the symmetric case treated in [Ma], in the spherical case there does not need to exist a maximal orbit in M mapping on a fixed orbit $Z \subset X_\delta$: this is shown by Example 6.5 and Example 7.2.

Let's give now a general lemma, answering whether or not a sublattice of a given lattice is saturated or not.

Lemma 5.9. *Let Λ be a lattice of rank n , freely generated by $\sigma_1, \dots, \sigma_n \in \Lambda$; let $\Gamma \subset \Lambda$ be a sublattice of rank m , freely generated $\gamma_1, \dots, \gamma_m \in \Lambda$. Write $\gamma_i = \sum_j a_{ij} \sigma_j$ and set $A = (a_{ij})$. Then Γ is saturated in Λ if and only if the greatest common divisor of the m -order minors of A is 1.*

Proof. If $J = \{j_1, \dots, j_m\} \subset \{1, \dots, n\}$ is a subset such that the matrix $A_J = (a_{ij_k})$ is not singular, set $\Gamma_J = \bigoplus_{k=1}^m \mathbf{Z} \sigma_{j_k}$ and consider the projection $\pi_J : \Gamma_{\mathbf{Q}} \rightarrow (\Gamma_J)_{\mathbf{Q}}$, which is an isomorphism by the assumption on J . Set $\overline{\Gamma} = \Gamma_{\mathbf{Q}} \cap \Lambda$ the saturation of Γ in Λ ; then $\overline{\Gamma} = \bigcap_J \pi_J^{-1}(\Gamma_J)$. Since

$$[\pi_J^{-1}(\Gamma_J) : \Gamma] = [\Gamma_J : \pi_J(\Gamma)] = \det(A_J),$$

then we get $[\overline{\Gamma} : \Gamma] = \gcd\{\det(A_J)\}_J$. \square

Consider now an orbit $Z \subset X_\delta$. Once it is known that $Z' = p^{-1}(Z)$ admits a wonderful completion (see Remark 4.2), then following Lemma is an easy consequence of Corollary 4.8 together with previous lemma:

Lemma 5.10. *An orbit $Z \subset X_\delta$ is not isomorphic to its corresponding orbit $Z' \subset \tilde{X}_\delta$ if and only if Z possesses a spherical root γ of the shape $\gamma = 2\sigma_1 + \dots + 2\sigma_k$, with $\sigma_1, \dots, \sigma_k \in \Sigma$.*

Therefore, as in Corollary 4.8, to any orbit $Z \subset X_\delta$ we can attach a subset $\Sigma(\delta_Z) \subset \Sigma_{Z'}$ whose elements are the spherical roots of Z' which have to be doubled to get the spherical roots of Z . If $\gamma \in \Sigma(\delta_Z)$, then either $\gamma \in S$ or $\gamma = \alpha_{i+1} + \dots + \alpha_{i+r}$ has support of type B_r or $\gamma = 2\alpha_{i+1} + \alpha_{i+2}$ has support of type G_2 .

Definition 5.11. If $\sigma \in \Sigma(G)$, then we say that:

- σ is of type B_r^I if $\sigma = \alpha_{i+1} + \dots + \alpha_{i+r}$ has support of type B_r ;
- σ is of type B_r^{II} if $\sigma = 2\alpha_{i+1} + \dots + 2\alpha_{i+r}$ has support of type B_r ;
- σ is of type G_2^I if $\sigma = 2\alpha_{i+1} + \alpha_{i+2}$ has support of type G_2 ;
- σ is of type G_2^{II} if $\sigma = 4\alpha_{i+1} + 2\alpha_{i+2}$ has support of type G_2 .

6. THE STRICT CASE.

Suppose that M is a strict wonderful variety. Denote $\Sigma B_2^I \subset \Sigma$ the set of spherical roots $\sigma = \alpha_\sigma^\# + \alpha_\sigma^b$ of type B_2^I (where $\alpha_\sigma^\#, \alpha_\sigma^b \in S$ are respectively the long simple root and the short simple root in the support of σ). Since M is spherically closed, both $\alpha_\sigma^\#$ and α_σ^b move exactly one color; set $\Delta(\alpha_\sigma^\#) = \{D^\#(\sigma)\}$ and $\Delta(\alpha_\sigma^b) = \{D^b(\sigma)\}$.

Lemma 6.1. *Let M be a strict wonderful variety and let δ be a faithful divisor on it; let $Z \subset X_\delta$ be an orbit. Then $Z \not\cong Z'$ if and only if there exist a spherical root $\gamma \in \Sigma_Z$ of type B_r^{II} and a spherical root $\sigma \in \text{Supp}_\Sigma(\gamma)$ of type B_2^I .*

Proof. Consider the minimal orbit $W_Z \subset M$ which maps on Z ; set $\Sigma(Z) = \{\sigma_1, \dots, \sigma_n\}$ and $\Sigma_Z = \{\gamma_1, \dots, \gamma_m\} \subset \mathbf{N}\Sigma$. Set $\gamma_i = \sum a_{ij}\sigma_j$ and consider the matrix $A(Z) = (a_{ij})$. Since $\Sigma \cap S = \emptyset$, by the explicit description of $\Sigma(G)$ we deduce that $0 \leq a_{ij} \leq 2$, for every i, j ; moreover, by the definition of $\Sigma(Z)$, we get that every column of $A(Z)$ possesses at least one non-zero entry. Since M possesses no simple roots, for every $\gamma \in \Sigma_Z$ there exists a spherical root $\sigma \in \text{Supp}_\Sigma(\gamma)$ such that $\sigma \notin \text{Supp}_\Sigma(\gamma')$ for every $\gamma' \in \Sigma_Z \setminus \{\gamma\}$. Therefore by Lemma 5.9 we get that $Z' \not\cong Z$ if and only if $A(Z)$ possesses a row whose unique entries are 0 and 2. Set $\gamma \in \Sigma_Z$ the corresponding spherical root. By the explicit description of $\Sigma(G)$, we deduce that γ must be either of type B_r^{II} or of type G_2^{II} ; moreover it is uniquely determined a spherical root $\sigma \in \text{Supp}_{\Sigma(Z)}(\gamma)$ which is of type $B_{r'}^I$ (with $r' \leq r$) in the first case and of type G_2^I in the second case. Since M is spherically closed, the latter cannot happen; therefore we are in the first case.

Suppose $r' > 2$ and denote $q : \text{Pic}(M) \rightarrow \text{Pic}(\overline{W}_Z)$ the pullback map. Since M and \overline{W}_Z are both spherically closed, α_σ^b moves a color $D^b(\sigma) \in \Delta$ and, as well, a color $'D^b(\sigma) \in \text{Supp}_{\Delta(W_Z)}(q(D^b(\sigma))) \subset \Delta(W_Z)$. Since $r' > 2$, we get $c(D^b(\sigma), \sigma') \geq 0$ for any spherical root $\sigma' \in \Sigma$: therefore it must be $D^b(\sigma) \in \text{Supp}_\Delta(\delta)$ and we get

$$'D^b(\sigma) \in \text{Supp}_{\Delta(W_Z)}(q(D^b(\sigma))) \subset \text{Supp}_{\Delta(W_Z)}(q(\delta)).$$

But then we get an absurd by part (i) of following lemma. □

Lemma 6.2. *Let M be a strict wonderful variety and let δ be a faithful divisor on it; let $\sigma \in \Sigma B_2^I$.*

- (i) *If $D^b(\sigma) \in \text{Supp}_\Delta(\delta)$, then no orbit $Z \subset X_\delta$ possesses a spherical root $\gamma \in \Sigma_Z$ of type B_r^{II} with $\sigma \in \text{Supp}_\Sigma(\sigma)$.*
- (ii) *If there exists $\sigma \in \Sigma B_2^I$ such that $\text{Supp}_\Delta(\delta) \cap \{D^\#(\sigma), D^b(\sigma)\} = \{D^\#(\sigma)\}$, then there exists an orbit $Z \subset X_\delta$ such that $2\sigma \in \Sigma_Z$; in particular the normalization morphism $p : \tilde{X}_\delta \rightarrow X_\delta$ is not bijective.*

Proof. (i). Suppose by absurd that $Z \subset X_\delta$ is such an orbit; set $Z \simeq G/K$ and $Z' \simeq G/K'$, with $K' \subset K$. Since K is the spherical closure of K' , the restriction $p : Z' \rightarrow Z$ induces a map $\Delta(Z') \rightarrow \Delta(Z)$ which is one-to-one; on the other hand, by Theorem 4.1 the map $\tilde{\phi}_{q(\delta)} : W_Z \rightarrow Z'$ induces an injection $\text{Supp}_{\Delta(W_Z)}(q(\delta)) \hookrightarrow \Delta(Z')$, therefore $\phi_{q(\delta)}$ too induces an injection $\text{Supp}_{\Delta(W_Z)}(q(\delta)) \hookrightarrow \Delta(Z)$. Consider now the color $\phi_{q(\delta)}('D^b(\sigma)) \in \Delta(Z)$, which is moved by the simple root α_σ^b : then we get an absurd since the existence of such a color is incompatible with the existence of a spherical root in Σ_Z of type B_r^Π supported on α_σ^b , such as γ is.

(ii). Consider the rank one orbit $W \subset M$ whose unique spherical root is σ and set $q : \text{Pic}(M) \rightarrow \text{Pic}(\overline{W})$ the pullback map. Set $\Delta(W)(\alpha_\sigma^b) = \{'D^b(\sigma)\}$ and $\Delta(W)(\alpha_\sigma^\sharp) = \{'D^\sharp(\sigma)\}$; then $\text{Supp}_{\Delta(W)}(q(\delta)) \cap \{'D^\sharp(\sigma), 'D^b(\sigma)\} = \{'D^\sharp(\sigma)\}$. Set $G/K \simeq Z = \phi_\delta(W)$ and $G/K' \simeq Z' = p^{-1}(Z)$, with $K' \subset K$; since $c('D^b(\sigma), \sigma) = 0$ and since $'D^\sharp(\sigma)$ is the unique color $'D \in \Delta(W)$ such that $c('D, \sigma) > 0$, we get that $\Lambda_{Z'} = \Lambda_W$ is freely generated by σ ; therefore K' is wonderful, with σ as unique spherical root. Since $D_\sigma^b \notin \text{Supp}_\Delta(\delta)$ maps dominantly on Z' , we get that $\Delta(Z')(\alpha_\sigma^b) = \emptyset$; therefore K' is not spherically closed and $2\sigma \in \Sigma_Z$. \square

Corollary 6.3. (i) *If M is a symmetric variety and if $\delta \in \text{Pic}(M)$ is any faithful divisor, then the normalization $p : \tilde{X}_\delta \rightarrow X_\delta$ is bijective.*
 (ii) *Suppose that the Dynkin diagram of G is simply-laced. If M is any strict wonderful variety for G and if $\delta \in \text{Pic}(M)$ is any faithful divisor, then the normalization $p : \tilde{X}_\delta \rightarrow X_\delta$ is bijective.*
 (iii) *If $D^b(\sigma) \in \text{Supp}_\Delta(\delta)$ for every $\sigma \in \Sigma B_2^I$, then the normalization morphism $p : \tilde{X}_\delta \rightarrow X_\delta$ is bijective.*

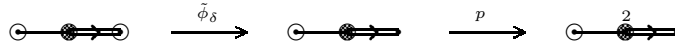
Proof. By the classification of symmetric varieties, we deduce that a symmetric variety never possesses a spherical root of type B_2^I . Then all of the claims above follow straightforward by previous lemma. \square

Another proof of Corollary 6.3 (i) was given in [Ma] with completely different methods. Following examples show some cases where the conditions of previous lemma are fulfilled:

Example 6.4. Consider the wonderful variety M whose spherical system is expressed by following spherical diagram



Number the simple roots from the left to the right; then the divisor $\delta = D_{\alpha_2}$ is faithful. Consider the codimension one orbit $W \subset M$ associated to the spherical root $\alpha_1 + \alpha_2$; following Proposition 5.1 and Corollary 4.8, we get the following sequence of spherical diagrams



where the first one represents the orbit $W \subset M$, the second one represents the orbit $\tilde{\phi}_\delta(W) \subset \tilde{X}_\delta$ and the third one represents the orbit $\phi_\delta(W) \subset X_\delta$.

Example 6.5. Consider the wonderful variety M whose spherical system is expressed by following spherical diagram

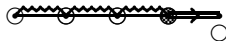


TABLE 1. Example 6.5 , $\delta = D_{\alpha_2}$.

| Maximal Orbits | Minimal Orbit | Orbit in \tilde{X}_δ | Orbit in X_δ | $\Sigma(\delta_Z)$ |
|-----------------------------------|---------------------|-----------------------------|---------------------|-----------------------------|
| $\{1, 2, 3, 4, 5\}$ | $\{1, 2, 3, 4, 5\}$ | | | \emptyset |
| $\{1, 2, 3, 4\}$ | $\{1, 2, 3, 4\}$ | | | \emptyset |
| $\{1, 2, 4, 5\}$ | $\{1, 2, 4\}$ | | | $\{\alpha_4 + \alpha_5\}$ |
| $\{1, 2, 3, 5\}$ | $\{1, 2\}$ | | | \emptyset |
| $\{2, 3, 4, 5\}$ | $\{2, 4\}$ | | | $\{\sum_{i=2}^5 \alpha_i\}$ |
| $\{1, 3, 4, 5\}$ $\{2, 3, 5\}$ | \emptyset | | | \emptyset |

Number the simple roots from the left to the right; then the divisor $\delta = D_{\alpha_2}$ is faithful. See Table 1 for a full list of the orbits in X_δ and in \tilde{X}_δ ¹.

As illustrated by previous examples, main examples of strict wonderful varieties possessing a faithful divisor δ such that the normalization morphism $p : \tilde{X}_\delta \rightarrow X_\delta$ is not bijective arise from the context of wonderful model varieties (see [L3]); as shown in the following the case of a general strict wonderful variety substantially follows from this particular case.

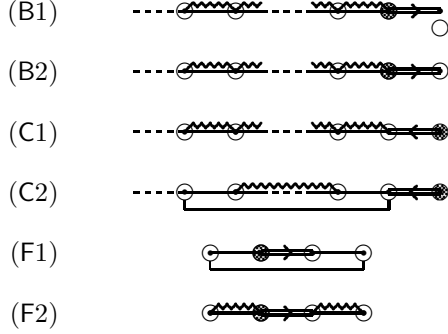
Consider a strict wonderful variety M and let δ be a faithful divisor on it. Let $\sigma \in \Sigma B_2^I$ be a spherical root of type B_2^I and set $\Gamma(\sigma)$ the connected component of the Dynkin diagram of G where σ is supported. If $\Gamma(\sigma)$ is of type B or C, number the simple roots in $\Gamma(\sigma)$ starting from the extreme where the double link is.

If $\{D_\sigma^b, D_\sigma^\sharp\}$ contains a distinguished subset of colors, then by Lemma 6.2 we get that there does not exist an orbit $Z \subset X_\delta$ possessing a spherical root γ of type $B_r^{\mathbb{I}}$ with $\sigma \in \text{Supp}_\Sigma(\gamma)$ if and only if $D_\sigma^b \in \text{Supp}_\Delta(\delta)$. For instance, this is the case if one of the following conditions is verified:

- $\Gamma(\sigma)$ is of type B or C and σ is the unique spherical root supported on α_2 ;
- $\Gamma(\sigma)$ is of type C and $2\alpha_2 \in \Sigma$;

This allows us to identify a “tail” in the spherical diagram of M containing σ as follows. Suppose in fact that $\{D_\sigma^b, D_\sigma^\sharp\}$ does not contain any distinguished subset of colors and suppose that $\Gamma(\sigma)$ is not of type F_4 (actually such case consists only of two subcases). Then by the above discussion there exists a spherical root supported on α_2 ; by a case by case check, it turns out that such root is either of type A_2 or, if $\Gamma(\sigma)$ is of type C, of type $A_1 \times A_1$. This implies that the spherical diagram of M in $\Gamma(\sigma)$ is of one of the following types:

¹For simplicity, in Table 1 and in Table 2 orbits in M are assigned by giving a subset of its spherical root index set.



Suppose that we are neither in case C2, nor in case F1 or F2 (such cases are easily treated directly); thus we are substantially reduced to the case of a wonderful model variety. Let $m(\sigma) \geq 3$ be such that $\alpha_{m(\sigma)}$ is the first simple root which is contained in the support of one and only one spherical root of type A_2 . For $1 \leq k \leq m(\sigma)$, set $\Delta(\alpha_k) = \{D_k\}$; set $\Delta(\sigma) = \{D_1, \dots, D_{m(\sigma)}\}$ and $\Delta(\sigma)^{even}, \Delta(\sigma)^{odd} \subset \Delta(\sigma)$ the subsets whose elements index is respectively even and odd. If they are defined, set

$$i(\delta, \sigma) = \min\{k \leq m(\sigma) : D_k \in \text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{even}\}$$

$$j(\delta, \sigma) = \min\{k \leq m(\sigma) : D_k \in \text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{odd}\}.$$

Lemma 6.6. *Let M be a strict wonderful variety possessing a spherical root σ of type B_2^I such that the spherical diagram of M in $\Gamma(\sigma)$ is of type B1; let δ be a faithful divisor on M . Then there does not exist any orbit $Z \subset X_\delta$ possessing a spherical root γ of type B_r^II with $\sigma \in \text{Supp}_\Sigma(\gamma)$ if and only if $D_1 \in \text{Supp}_\Delta(\delta)$ or if following conditions are both satisfied:*

- (i) $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{even} = \emptyset$;
- (ii) *If there exists a spherical root $\sigma' \in \Sigma$ supported on $\alpha_{m(\sigma)+1}$, then $m(\sigma)$ is odd.*

Proof. By Lemma 6.2, we may assume that $\text{Supp}_\Delta(\delta) \cap \{D_1, D_2\} = \emptyset$. Observe that $\Delta(\sigma) \setminus \{D_{m(\sigma)}\}$ is a distinguished subset of colors and that, conversely, any distinguished subset of colors which intersects $\Delta(\sigma)$ contains $\Delta(\sigma) \setminus \{D_{m(\sigma)}\}$.

If $W \subset M$ is an orbit with spherical roots set $\Sigma' \subset \Sigma$ and colors set Δ' , set $\Delta'(\alpha_i) = \{D'_i\}$ for $1 \leq i \leq m(\sigma)$ and $\Delta'(\sigma) = \{D'_1, \dots, D'_{m(\sigma)}\}$. Denote $q : \text{Pic}(M) \rightarrow \text{Pic}(\overline{W})$ the pullback map and observe that q induces a bijection between $\Delta(\sigma)$ and $\Delta'(\sigma)$. More precisely, $q(D_i) = D'_i$ for every $i > 1$, while

$$q(D_1) = \begin{cases} D'_1 & \text{if } 2\alpha_1 \in \Sigma' \\ 2D'_1 & \text{if } 2\alpha_1 \notin \Sigma' \end{cases} :$$

thus, if $i \leq m(\sigma)$, then δ is supported on D_i if and only if $q(\delta)$ is supported on D'_i .

(\Leftarrow) Suppose that $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{even} = \emptyset$. Set $M' \subset M$ the G -stable prime divisor associated to the spherical root $2\alpha_1$ and set $W \subset M'$ its open orbit. If $Z \subset X_\delta$ is an orbit possessing a spherical root γ of type B_r^II with $\sigma \in \text{Supp}_\Sigma(\gamma)$, then $\sigma \in \Sigma(Z)$ implies that $2\alpha_1 \notin \Sigma(Z)$. Therefore by Lemma 5.3 such an orbit is necessarily contained in $\phi_\delta(M')$ and, in order to prove the claim, it is enough to show that it is true for any orbit which is contained in $\phi_\delta(M')$.

Set $\Delta^* \subset \Delta'$ the maximal distinguished subset of colors which does not intersect the support of $q(\delta)$. If there is no spherical root supported on $\alpha_{m(\sigma)+1}$ or if $m(\sigma)$ is odd, then $\Delta'(\sigma)^{even}$ is distinguished; by $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{even} = \emptyset$ we deduce then that $\Delta'(\sigma)^{even} \subset \Delta^*$. If $\Delta^* \cap \Delta'(\sigma)^{odd} \neq \emptyset$, then it should be $\Delta'(\sigma) \setminus \{D'_{m(\sigma)}\} \subset \Delta^*$, which contradicts the faithfulness of δ . This implies that $\Delta^* \cap$

$\Delta'(\sigma) = \Delta'(\sigma)^{even}$, which in turn implies that $\sigma \notin \Sigma(\phi_\delta(W))$. To conclude, it is enough to observe that, if $Z \subset \phi_\delta(M')$ is any orbit, then $\Sigma(Z) \subset \Sigma(\phi_\delta(W))$.

(\implies) Consider the codimension one orbit W whose spherical roots set is $\Sigma' = \Sigma \setminus \{\alpha_2 + \alpha_3\}$; set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. Denote $\Delta^* \subset \Delta'$ the maximal distinguished subset of colors which does not intersect the support of the pullback divisor $q(\delta)$. Since it is non-negative on any spherical root in Σ' , we obtain $D'_1 \in \Delta^*$. In order to show that $2\sigma \in \Sigma_Z$, then it's enough to show that $D'_2 \notin \Delta^*$. This implies $\sigma \in \Lambda_{Z'}$: in fact on one hand $D'_3 \in \Delta^*$ implies $D'_2 \in \Delta^*$, while on the other hand $c(D', \sigma) = 0$ for every $D' \in \Delta' \setminus \{D'_2, D'_3\}$. But $D'_1 \in \Delta^*$ implies that $\Delta(Z')(\alpha_1) = \Delta(Z)(\alpha_1) = \emptyset$, which in turn implies that $\sigma \notin \Lambda_Z$. Therefore, if $D'_2 \notin \Delta^*$, then $\sigma \in \Lambda_{Z'} \setminus \Lambda_Z$ and $2\sigma \in \Sigma_Z$.

Suppose first that $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{even} \neq \emptyset$. If $D'_2 \in \Delta^*$, then it must be either $\Delta'(\sigma) \setminus \{D'_{m(\sigma)}\} \subset \Delta^*$ or $\Delta'(\sigma)^{even} \subset \Delta^*$. Since δ is faithful, the first case is not possible; on the other hand, we assumed $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{even} \neq \emptyset$, thus the second case neither is possible. Therefore $D'_2 \notin \Delta^*$.

Suppose now that $m(\sigma)$ is even and that there exists a spherical root $\sigma' \in \Sigma'$ supported on $\alpha_{m(\sigma)+1}$. Set $m_1 := m(\sigma)$ and observe that $\sigma' = \alpha_{m_1+1} + \dots + \alpha_{m_1+r}$ has necessarily support of shape A_r , with $r \geq 2$; since the table which defines the Cartan pairing is the same, observe that we may assume, for simplicity, that σ' is a spherical root with support of shape A_2 . Set $m_2 > m_1 + 1$ the first integer such that α_{m_2} is in the support of exactly one spherical root of type A_2 . Proceed similarly, “shrinking” every possible spherical root with support of type A_r with $r > 2$ to a spherical root with support of type A_2 , and define a sequence

$$m_k > m_{k-1} + 1 > \dots > m_1 + 1$$

until no spherical root is supported on α_{m_k+1} . If $1 \leq j \leq m_k$, set $\Delta(\alpha_j) = \{D_j\}$ and $\Delta'(\alpha_j) = \{D'_j\}$; if $1 \leq i \leq k$, set $\Delta_i = \bigcup_{j=m_{i-1}+1}^{m_i} \Delta(\alpha_j)$ and $\Delta'_i = \bigcup_{j=m_{i-1}+1}^{m_i} \Delta'(\alpha_j)$ (where $m_0 := 0$). Set moreover $\Delta_i^{even} \subset \Delta_i$ and $(\Delta'_i)^{even} \subset \Delta'_i$ the subsets whose elements index is even. Define $k_0 \in \{1, \dots, k\}$ the first integer such that m_{k_0} is odd or set $k_0 = k$ otherwise. Then it's easy to show that $D'_2 \in \Delta^*$ if and only if $\Delta^* \cap \Delta'_i = (\Delta'_i)^{even}$, for every $i \leq k_0$: since $(\Delta_{k_0})^{even} \subset \Delta$ is distinguished, this is impossible. \square

Corollary 6.7. *Let M be a strict wonderful variety possessing a spherical root σ of type B_2^1 such that the spherical diagram of M in $\Gamma(\sigma)$ is of type B2; let δ be a faithful divisor on M . Then there does not exist any orbit $Z \subset X_\delta$ possessing a spherical root γ of type B_r^Π with $\sigma \in \text{Supp}_\Sigma(\gamma)$ if and only if $D_1 \in \text{Supp}_\Delta(\delta)$.*

Proof. Let M' be the wonderful variety whose spherical system is the same one of M with one further spherical root $2\alpha_1$: then M is identified with a G -stable prime divisor of M' and the spherical diagram of M' in $\Gamma(\sigma)$ is of the type considered in previous lemma. Denote Σ' and Δ' the set of spherical roots and the set of colors of M' ; observe that the pullback map $q : \text{Pic}(M') \rightarrow \text{Pic}(M)$ induces an isomorphism between the sublattices generated by $\Delta \setminus \{D_{\alpha_1}\}$ and $\Delta' \setminus \{D'_{\alpha_1}\}$. If $D_1 \in \text{Supp}_\Delta(\delta)$ then the claim follows straightforward; thus we may assume $D_1 \notin \text{Supp}_\Delta(\delta)$ and we may identify δ with a divisor δ' on M' which is still faithful.

If $Z \subset \phi_{\delta'}(M')$ is an orbit possessing a spherical root γ of type B_r^Π with $\sigma \in \text{Supp}_\Sigma(\gamma)$, then $2\alpha_1 \notin \text{Supp}_{\Sigma'}(\gamma)$ and by Proposition 5.3 we get $Z \subset X_\delta \subset \phi_\delta(M')$: therefore such an orbit exists in X_δ if and only if it exists in $\phi_{\delta'}(M')$ and we can apply previous lemma. In order to get the claim, it's enough to observe that (with the same notations of previous lemma) if $m(\sigma)$ is odd, then $\Delta(\sigma)^{even} = q(\Delta'(\sigma)^{even}) \subset \Delta$ is a distinguished subset; therefore $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{even} \neq \emptyset$ and consequently $\text{Supp}_{\Delta'}(\delta') \cap \Delta'(\sigma)^{even} \neq \emptyset$ as well. \square

Lemma 6.8. *Let M be a strict wonderful variety possessing a spherical root σ of type B_2^I such that the spherical diagram of M in $\Gamma(\sigma)$ is of type C1; let δ be a faithful divisor on M . Then there does not exist any orbit $Z \subset X_\delta$ possessing a spherical root γ of type B_r^{II} with $\sigma \in \text{Supp}_\Sigma(\gamma)$ if and only if following conditions are both satisfied*

- (i) $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{even}} \neq \emptyset$;
- (ii) If $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{odd}} \neq \emptyset$, then $j(\delta, \sigma) \geq i(\delta, \sigma) - 1$.

Proof. Set $i := i(\delta, \sigma)$ and $j := j(\delta, \sigma)$. Observe that at least one between $\Delta(\sigma)^{\text{even}}$, $\Delta(\sigma)^{\text{odd}}$ is distinguished: therefore at least one between i and j is well defined. By Lemma 6.2, we may suppose $\min\{i, j\} > 2$. Number the $m(\sigma) - 1$ spherical roots supported on $\{\alpha_1, \dots, \alpha_{m(\sigma)}\}$ from the right to left.

If $W \subset M$ is an orbit with spherical roots set $\Sigma' \subset \Sigma$ and colors set Δ' , set $\Delta'(\alpha_k) = \{D'_k\}$ for $1 \leq k \leq m(\sigma)$ and $\Delta'(\sigma) = \{D'_1, \dots, D'_{m(\sigma)}\}$. Denote $q : \text{Pic}(M) \rightarrow \text{Pic}(\overline{W})$ the pullback map and observe that q induces a bijection between $\Delta(\sigma)$ and $\Delta'(\sigma)$. Since $q(D_k) = D'_k$ for every $k \leq m(\sigma)$, δ is supported on D_k if and only if $q(\delta)$ is supported on D'_k .

(\implies) Suppose that j is defined and, in case i is defined too, suppose that $j < i - 1$. Consider the orbit $W \subset M$ whose spherical roots are $\sigma_1, \dots, \sigma_j$; set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. Then the maximal distinguished subset of colors $\Delta^* \subset \Delta$ which does not intersect the support of $q(\delta)$ is

$$\Delta' \setminus (\Delta'(\sigma)_{\leq j+2}^{\text{odd}} \cup \text{Supp}_{\Delta'}(q(\delta))),$$

which by hypothesis contains $\Delta'(\sigma)_{\leq j+1}^{\text{even}}$ (where the notations are the obvious ones). Thus $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$; since $c(D', \sigma) = 0$ for every $D' \in \Delta' \setminus \{D'_1, D'_3\}$, by Proposition 5.1 we get $\sigma \in \Lambda_{Z'}$. On the other hand, $D'_2 \in \Delta^*$ implies $\Delta(Z)(\alpha_2) = \emptyset$: since Z is spherically closed, we get then $\sigma \notin \Sigma_Z$ and $2\sigma \in \Sigma_Z$.

(\impliedby) Suppose i is defined and, in case j is defined too, suppose that $j \geq i - 1$. Fix an orbit $W \subset M$ with spherical roots set Σ_W and colors set Δ' ; set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. If $\sigma \notin \Sigma_W$, then there is nothing to prove; thus we may assume $\sigma \in \Sigma_W$. Set $\Delta^* \subset \Delta'$ the maximal distinguished subset of colors which does not intersect the support of $q(\delta)$; observe that $2\sigma \in \Sigma_Z$ if and only if $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$. Such condition does not hold if $\sigma_2 \notin \Sigma_W$ or if $\sigma_3 \notin \Sigma_W$: thus we may assume that $\Sigma_W \supset \{\sigma_1, \sigma_2, \sigma_3\}$. Set $k < m(\sigma)$ the maximum integer such that $\sigma_t \in \Sigma_W$ for every $t \leq k$: localizing with respect to the subset of spherical roots $\{\sigma_1, \dots, \sigma_k\}$ and considering its distinguished subsets of colors, we get that, if $D'_2 \in \Delta^*$, then either $\Delta'(\sigma)_{\leq k} \subset \Delta^*$ or $\Delta'(\sigma)_{\leq k+1}^{\text{even}} \subset \Delta^*$. If we are in the first case, then we are done; suppose we are in the second case. Then it must be $i > k + 1$ and, by the hypothesis, we get $j \geq k + 1$: thus $\Delta'(\sigma)_{\leq k}$ is a distinguished subset which does not intersect the support of $q(\delta)$. Therefore the condition $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$ is never fulfilled, which shows the claim. \square

Combining Lemma 6.6, Corollary 6.7 and Lemma 6.8, we get the following theorem. The cases wherein the spherical diagram of M in $\Gamma(\sigma)$ is of type C2, F1 or F2 are easily treated case by case.

Theorem 6.9. *Let M be a strict wonderful variety and δ a faithful divisor on it. Then the normalization morphism $p : \tilde{X}_\delta \rightarrow X_\delta$ is bijective if and only if the following conditions are fulfilled, for any spherical root $\sigma \in \Sigma B_2^I$:*

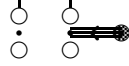
- (i) *If the spherical diagram of M in $\Gamma(\sigma)$ is of type B1, then $D^\flat(\sigma) \in \text{Supp}_\Delta(\delta)$ or following conditions are both fulfilled:*
 - $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{even}} = \emptyset$;
 - *If there exists a spherical root $\gamma \in \Sigma$ supported on $\alpha_{m(\sigma)+1}$, then $m(\sigma)$ is odd.*

- (ii) If the spherical diagram of M in $\Gamma(\sigma)$ is of type B2, then $D^b(\sigma) \in \text{Supp}_\Delta(\delta)$.
- (iii) If the spherical diagram of M in $\Gamma(\sigma)$ is of type C1, then following conditions are both fulfilled
 - $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{even}} \neq \emptyset$;
 - If $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{odd}} \neq \emptyset$, then $j(\delta, \sigma) \geq i(\delta, \sigma) - 1$.
- (iv) Otherwise, if $D^\sharp(\sigma) \in \text{Supp}_\Delta(\delta)$, then $D^b(\sigma) \in \text{Supp}_\Delta(\delta)$ as well.

7. THE NON-STRICT CASE.

Suppose that M is a non-strict wonderful variety and let $\delta = \sum_\Delta n(\delta, D)D$ be a faithful divisor on M . Suppose that $Z \subset X_\delta$ is an orbit such that $\Sigma(\delta_Z)$ contains a non-simple spherical root γ . Following examples show that, unlike from the strict case (Lemma 6.1), it may be as well γ of type G_2^I and, in case γ is of type B_r^I , then it does not necessarily come from a spherical root $\sigma \in \Sigma$ of type B_2^I .

Example 7.1. Consider the wonderful variety M whose spherical system is expressed by following spherical diagram

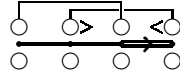


Number the simple roots from the left to the right; then the divisor $\delta = D_{\alpha_1}^+$ is faithful. Consider the codimension one orbit $W \subset M$ whose spherical roots are α_2 and $\alpha_2 + \alpha_3$; following Proposition 5.1 and Corollary 4.8, we get the following sequence of spherical diagrams



where the first one represents the orbit $W \subset M$, the second one represents the orbit $\tilde{\phi}_\delta(W) \subset \tilde{X}_\delta$ and the third one represents the orbit $\phi_\delta(W) \subset X_\delta$.

Example 7.2. Consider the wonderful variety M whose spherical system is expressed by following spherical diagram



Number the simple roots from the left to the right; then the divisor $\delta = D_{\alpha_1}^+$ is faithful. See Table 2 for a full list of the orbits in \tilde{X}_δ and in X_δ .

Lemma 7.3. Let M be a spherically closed wonderful variety and let $\delta = \sum_\Delta n(\delta, D)D$ be a faithful divisor on it; let $\alpha \in S \cap \Sigma$ be a simple spherical root.

- (i) If $Z \subset X_\delta$ is an orbit such that $2\alpha \in \Sigma_Z$, then $n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)$.
- (ii) If $n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)$ is non-zero, then there exists an orbit $Z \subset X_\delta$ such that $2\alpha \in \Sigma_Z$.

Proof. If $M' \subset M$ is a localization with spherical roots set $\Sigma' \subset \Sigma$, let $q : \text{Pic}(M) \rightarrow \text{Pic}(M')$ be the pullback map and consider the commutative diagram

$$\begin{array}{ccc} \text{Pic}(M) & \xrightarrow{q} & \text{Pic}(M') \\ & \searrow \omega & \downarrow \omega' \\ & & \mathcal{X}(B) \end{array}$$

where ω and ω' are the restrictions of linear bundles to the closed orbit. Set Δ and Δ' respectively the set of colors of M and of M' ; if $\alpha \in S \cap \Sigma$, set $\Delta(\alpha) = \{D_\alpha^+, D_\alpha^-\}$,

TABLE 2. Example 7.2, $\delta = D_{\alpha_1}^+$.

| Maximal Orbits | Minimal Orbit | Orbit in \tilde{X}_δ | Orbit in X_δ | $\Sigma(\delta_Z)$ |
|-----------------------------|------------------|-----------------------------|---------------------|---------------------------|
| $\{1, 2, 3, 4\}$ | $\{1, 2, 3, 4\}$ | | | \emptyset |
| $\{1, 2, 3\}$ | $\{1, 2, 3\}$ | | | \emptyset |
| $\{1, 3, 4\}$ | $\{1, 3, 4\}$ | | | $\{\alpha_4\}$ |
| $\{2, 3, 4\}$ | $\{2, 3, 4\}$ | | | \emptyset |
| $\{1, 3\}$ | $\{1, 3\}$ | | | \emptyset |
| $\{3, 4\}$ | $\{3, 4\}$ | | | $\{\alpha_3 + \alpha_4\}$ |
| $\{1, 2, 4\}$ $\{2, 3\}$ | \emptyset | | | \emptyset |

while if $\alpha \in S \cap \Sigma'$ set $\Delta'(\alpha) = \{D_\alpha^+, D_\alpha^-\}$. If $\alpha \in S \cap \Sigma'$, then by the equality

$$\omega'(q(D_\alpha^+)) = \omega(D_\alpha^+) = \sum_{\beta \in S: D_\alpha^+ \in \Delta(\beta)} \omega_\beta$$

we get that $q(D_\alpha^+)$ is supported on one and only one color in $\Delta'(\alpha)$ with multiplicity 1. Therefore $n(q(\delta), D_\alpha^+) = n(\delta, D_\alpha^+)$ and similarly $n(q(\delta), D_\alpha^-) = n(\delta, D_\alpha^-)$.

(i). Let $Z \subset X_\delta$ be an orbit possessing 2α as a spherical root; let $Z' = p^{-1}(Z)$ and let Z'' be the spherical closure of Z' , which still maps on Z . Since the projection $Z' \rightarrow Z''$ identifies the respective sets of colors, we have a natural isomorphism $\text{Pic}(M(Z')) \simeq \text{Pic}(M(Z''))$; by Theorem 4.1, we may identify then $\text{Pic}(M(Z''))$ with the sublattice of $\text{Pic}(\bar{W})$ generated by the classes of colors which do not map surjectively on Z and this shows that α is a spherical root for Z'' . Set $\delta_Z \in \text{Pic}(M(Z''))$ the pullback of the hyperplane bundle on \bar{Z} : then by Corollary 4.8 it follows that $n(\delta_Z, D_\alpha^+) = n(\delta_Z, D_\alpha^-)$, which by the discussion at the beginning implies $n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)$.

(ii). If $W \subset M$ is any orbit and if $D \in \Delta$ is any color, then $n(\delta, D) \neq 0$ implies that the pullback divisor $q(D) \in \text{Pic}(\bar{W})$ maps non-surjectively on $\phi_\delta(W) \subset X_\delta$. Therefore we can apply the same argument used before to the rank one orbit W whose unique (loose) spherical root is α and we obtain that the orbit $\phi_\delta(W)$ possesses 2α as a spherical root. \square

As shown by Example 7.2, if $\alpha \in S \cap \Sigma$ is such that $n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-) = 0$, then it may not exist any orbit $Z \subset X_\delta$ possessing 2α as a spherical root.

As a corollary of previous lemma, we get the following sufficient conditions.

Corollary 7.4. *Let M be a spherically closed wonderful variety and let $\delta = \sum_{\Delta} n(\delta, D)D$ be a faithful divisor on it.*

- (i) *If there exists $\alpha \in S \cap \Sigma$ such that $n(\delta, D_{\alpha}^{+}) = n(\delta, D_{\alpha}^{-}) \neq 0$, then the normalization morphism $p : \tilde{X}_{\delta} \rightarrow X_{\delta}$ is not bijective.*
- (ii) *If the Dynkin diagram of G is simply-laced and if $n(\delta, D_{\alpha}^{+}) \neq n(\delta, D_{\alpha}^{-})$ for every $\alpha \in S \cap \Sigma$, then the normalization morphism $p : \tilde{X}_{\delta} \rightarrow X_{\delta}$ is bijective.*

Reasoning as in Lemma 6.2 and in Corollary 6.3, other similar sufficient conditions can be obtained imposing conditions on the support of the divisor δ near the double links of the Dynkin diagram of G .

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Current address: Dipartimento di Matematica “Guido Castelnuovo”, “Sapienza” Università di Roma, Piazzale Aldo Moro 5, 00185 Roma, Italy

E-mail address: gandini@mat.uniroma1.it; jacopo.gandini@gmail.com